

Calogero-Moser systems and Hitchin systems

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ABSTRACT: We exhibit the elliptic Calogero-Moser system as a Hitchin system of G -principal Higgs pairs. The group G , though naturally associated to any root system, is not semi-simple. We then interpret the Lax pairs with spectral parameter of [dP1] and [BSC1] in terms of equivariant embeddings of the Hitchin system of G into that of $GL(N)$.

1. Introduction

The Calogero-Moser Hamiltonian system must be one of the most thoroughly studied Hamiltonian systems, yet many aspects of its geometry remain quite mysterious.

One can associate Calogero-Moser systems to any root system \mathcal{R} on the Lie algebra \mathfrak{h} of a torus H of dimension r and to any elliptic curve Σ (as well as to the limiting cases of rational nodal or cusp curves, the “rational” and “trigonometric” cases respectively). In canonical coordinates $(x, p) \in \mathfrak{h} \times \mathfrak{h}^*$ the system is very simple, and is given by the Hamiltonian

$$CM = p \cdot p + \sum_{\alpha \in \mathcal{R}} m_{|\alpha|} \mathfrak{p}(\alpha(x)). \quad (1.1)$$

where $\mathfrak{p}(x)$ is the Weierstrass \mathfrak{p} -function, and the $m_{|\alpha|}$ are constants depending only on the norm of the root α . In the rational case, one replaces \mathfrak{p} by the function x^{-2} , and in the trigonometric case, by the function $\sin(x)^{-2}$.

The system was of course obtained in a step by step fashion from the rational and trigonometric $Sl(n)$ case, by various people (for a survey, see [OP]), who in particular noticed that one could replace the linear functions $(x_i - x_j)$ occurring in the original $SL(N)$ case by the roots of more general root systems, while maintaining integrability. The presence of root systems naturally suggests that the Calogero-Moser systems have some geometric origin, tied to Lie groups.

In particular, when one is discussing ties between Lie groups or algebras and integrable systems, one is immediately led to look for Lax pairs $\dot{L} = [M, L]$, and indeed this is the way much of the work on the Calogero Moser system has progressed, e.g. with [OP, K] and more recently [dP1, dP2, dP3, BCS1, BCS2], so that one now has Lax pairs with a spectral parameter for all of the Calogero-Moser systems. Nevertheless, there are several mysterious aspects to many of these Lax pairs, and in general, there is a lack of concordance between the Lax pairs and the geometry of the group:

- The first is that, for the most part, the Lax matrices L are not in the Lie algebra of the root systems, though they occasionally occur in a symmetric space construction associated to the Lie algebra [OP]. Often, however, they are in some $Gl(N)$, where N is not even a dimension of a non-trivial representation of the group.

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- As was pointed out in [Do], while there is a manifest invariance of the Calogero-Moser Hamiltonian under the action of the Weyl group, there is on the other hand, for most semisimple Lie groups, no Weyl invariant coadjoint orbits. Such orbits would usually be an essential ingredient for a suitable geometric version of the Calogero-Moser systems. Also, in general, there are no orbits of dimension twice the dimension of the torus, which one would also want.
- Finally, one has a Calogero-Moser system for the root systems BC_n , and these do not even correspond to groups.

There is one case for which a satisfactory geometric version of the Calogero-Moser system exists, that of $SL(N, \mathbb{C})$ (see, e.g. [Do]). In this case, one finds that the Calogero-Moser system is a generalized Hitchin system over a moduli space of stable pairs over the elliptic curve. That is, one considers the moduli space of pairs (E, ϕ) , where

- E is a rank N degree 0 bundle on Σ with trivial determinant, and
- ϕ is a section of $End(E) \otimes K_\Sigma$ with a simple pole at the origin whose residue is a conjugate of $m \cdot diag(1, 1, 1, \dots, 1, -N + 1)$.

This realises the phase space in a natural way: the Hamiltonians are the coefficients of the equation of the “spectral curve” of ϕ ; and one has natural compactifications of the level sets of the Calogero-Moser hamiltonians as Jacobians of the spectral curves.

The following note attempts to explain some of the geometry of the Calogero-Moser systems for an arbitrary root system \mathcal{R} in terms of the geometry of a modified Hitchin system. The departure from previous work is that we do not use as structure group the semi-simple group associated to the root system, but rather a group which one can construct for any root system, whose connected component of the identity is the semi-direct product of the torus with the sum of the root spaces. The Weyl group acts on these, and this allows one to construct in a natural way for any root system some Weyl invariant coadjoint orbits of the correct dimension, to which one can associate Hitchin systems, over which the Calogero Moser Hamiltonian appears naturally. The Lax pairs with spectral parameter of [dP1, BCS2] appear in a natural way from embeddings of the Lie algebras of our groups into $Gl(V)$, where V is a sum of weight spaces invariant under the Weyl group W of the root system. These embeddings are not homomorphisms, but are invariant under the torus and the Weyl group action.

While this in some way clears up some of the mystery surrounding the Calogero-Moser systems, and in particular addresses the three facts outlined above, there are several aspects that are still unexplained: the first is that while the Calogero-Moser Hamiltonian occurs naturally, the other commuting Hamiltonians have no easy interpretation in our setup and only seem to occur naturally after embedding into the Hitchin system for $Gl(V)$. The other remaining question is understanding the compactifications of the level sets of the Hamiltonians (as Abelian varieties). This would require an enlargement of our phase space.

Section 2 of this paper is devoted to recalling certain facts about elliptic curves; section

3 is similarly devoted to the required facts on generalized Hitchin systems. In section 4, after discussing extensions of the Weyl group by the torus, we introduce the group which interests us, and discuss its properties. The next section is devoted to Hitchin systems with this group as structure group, and we show how the Calogero-Moser systems arise. The sixth section is devoted to a discussion of how this systems embeds into the Hitchin systems for $Gl(V)$, giving the Lax pairs of [dP1], [BCS2].

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2. Line bundles on an elliptic curve

Let Λ be a non-degenerate lattice in \mathbb{C} with generators $2\omega_1, 2\omega_2$ and let $\Sigma = \mathbb{C}/\Lambda$ be the corresponding elliptic curve. Denote the origin by p_0 . We have on \mathbb{C} the standard elliptic functions $\sigma(z), \zeta(z)$ with expansions at $z = 0$

$$\begin{aligned}\sigma(z) &= z + O(z^5) \\ \zeta(z) &= \frac{1}{z} + O(z^3),\end{aligned}\tag{2.1}$$

and periodicity relations

$$\begin{aligned}\sigma(z + 2\omega_i) &= -\sigma(z) \exp(2\eta_i(z + 2\omega_i)), \\ \zeta(z + 2\omega_i) &= \zeta(z) + 2\eta_i,\end{aligned}\tag{2.2}$$

with $\eta_i = \zeta(\omega_i)$. We have

$$\frac{d}{dz} \log(\sigma(z)) = \zeta(z), \quad \frac{d}{dz} \zeta(z) = -\mathbf{p}(z),$$

where $\mathbf{p}(z)$ is the standard Weierstrass \mathbf{p} -function. From the periodicity relations, one has that the function

$$\rho^1(x, z) = \frac{\sigma(z - x)}{\sigma(z)\sigma(x)} e^{x\zeta(z)}\tag{2.3}$$

is well defined on the elliptic curve with parameter z , with an essential singularity at the origin, and a single zero at $x = z$. If we set

$$\rho^0(x, z) = e^{-x\zeta(z)} \rho^1(x, z),\tag{2.4}$$

we find that ρ^0 has a single pole in z at the origin. Covering Σ by $U_1 = \Sigma - (\text{origin})$, $U_0 = \text{disk around origin}$, we can reinterpret the relation (2.4) as saying that ρ^1, ρ^0 define a section of the line bundle L_x with transition function $e^{-x\zeta(z)}$; this section has a single pole at the origin. Let p_x be the point $z = x$ on Σ ; L_x corresponds to the divisor $p_x - p_0$.

There is another way of representing sections of the line bundle L_x , which is as functions f on \mathbb{C} satisfying automorphy relations:

$$f(z + 2\omega_i) = f(z) \exp(-2\eta_i x).\tag{2.5}$$

In this way, the function $\rho^0(x, z)$ also represents a section of L_x with a simple pole at the origin.

For use later on, we note that

$$\zeta(z)\rho^0(x, z) = e^{-x\zeta(z)} \frac{\partial \rho^1(x, z)}{\partial x} - \frac{\partial \rho^0(x, z)}{\partial x}. \quad (2.6)$$

As a function of z , we have expansions

$$\begin{aligned} \rho^0(x, z) &= \frac{-1}{z} + \frac{\sigma'(x)}{\sigma(x)} + O(z), \\ \frac{d}{dx}\rho^0(x, z) &= \frac{d^2}{dx^2}(\log(\sigma(x))) + O(z), \end{aligned} \quad (2.7)$$

We have

$$\rho^0(x, z)\rho^0(-x, z) = \rho^1(x, z)\rho^1(-x, z) = \mathfrak{p}(z) - \mathfrak{p}(x). \quad (2.8)$$

Finally, consider

$$\begin{aligned} I_n : \Sigma &\rightarrow \Sigma \\ z &\mapsto nz, \end{aligned} \quad (2.9)$$

and let I_n^* denote the induced action on line bundles; we note that $I_n^*(L_x) = L_{nx}$. The pull-back $I_n^*(\rho(x, z)) = \rho(x, nz)$ has automorphy factors $\exp(-2n\eta_i x)$, and so one can represent a section by $\rho^0(x, nz)$ in the trivialization given above:

$$\rho^0(x, nz) = e^{-nx\zeta(z)} \hat{\rho}^1(x, z) \quad (2.10)$$

for a suitable function $\hat{\rho}^1$ on U_1 ; one has

$$n\zeta(z)\rho^0(x, nz) = e^{-nx\zeta(z)} \frac{\partial \hat{\rho}^1}{\partial x}(x, nz) - \frac{\partial \rho^0}{\partial x}(x, nz). \quad (2.11)$$

3. Generalised Hitchin systems on an elliptic curve

Let G be a complex Lie group. Following [Ma], we will consider the moduli space \mathcal{M}_G of pairs

$$(G\text{-bundle } P_G \text{ on } \Sigma, \text{ trivialisation } tr \text{ of } P_G \text{ at } p_0)$$

and its cotangent bundle $T^*\mathcal{M}_G$. For $P_G \in \mathcal{M}_G$, let $P_{\mathfrak{g}}$ be the adjoint bundle associated to P_G , and $P_{\mathfrak{g}^*}$ the associated coadjoint bundle. For any vector bundle V , set $V(p_0) = V \otimes \mathcal{O}(p_0)$, $V(-p_0) = V \otimes \mathcal{O}(-p_0)$. The fibre of $T^*\mathcal{M}_G \rightarrow \mathcal{M}_G$ at P_G is the vector space $H^0(\Sigma, P_{\mathfrak{g}^*}(p_0))$, (the canonical bundle of Σ is trivial) so that the $T^*\mathcal{M}_G$ is a space of triples

$$(G\text{-bundle } P_G \text{ on } \Sigma, \text{ trivialisation } tr \text{ at } p_0, \text{ section } \phi \text{ of } P_{\mathfrak{g}^*} \text{ with a pole at } p_0)$$

Dually, the tangent space to \mathcal{M}_G is $H^1(\Sigma, P_{\mathfrak{g}}(-p_0))$. We have, at (P_G, T) , the exact sequence for $T(T^*\mathcal{M}_G)$:

$$0 \rightarrow H^0(\Sigma, P_{\mathfrak{g}^*}(p_0)) \rightarrow T(T^*\mathcal{M}_G) \rightarrow H^1(\Sigma, P_{\mathfrak{g}}(-p_0)) \rightarrow 0. \quad (3.1)$$

The group G acts naturally on the trivialisations, and so acts symplectically on $T^*\mathcal{M}_G$. The moment map for this action is simply the residue of ϕ at p_0 , expressed in the trivialisation tr . One can take symplectic reductions of $T^*\mathcal{M}_G$ under this action, and obtain reduced moduli spaces $(T^*\mathcal{M}_G)_{\text{red}}$.

Let F be a homogeneous invariant function on \mathfrak{g}^* of degree n , and ω an element of $H^1(\Sigma, K_{\Sigma}^{-n+1}(-np_0)) \simeq H^1(\Sigma, \mathcal{O}(-np_0))$. Applying F to ϕ , one obtains an element $F(\phi)$ of $H^0(\Sigma, \mathcal{O}(np_0))$, and so one can define the Hamiltonian F_{ω} on $T^*\mathcal{M}_G$ by $F_{\omega}(P_G, \phi) = \{F(\phi), \omega\}$ where $\{, \}$ denotes the Serre duality pairing. These Hamiltonians descend to the reduced spaces $(T^*\mathcal{M}_G)_{\text{red}}$, and define there an integrable system, the generalized Hitchin system.

Explicit formulae

We can cover Σ by opens U_0, U_1 as above, and choose trivialisations on these opens, with the one on U_0 compatible with the trivialisation tr . Let $T = T_{1,0}$ be the corresponding transition function $U_0 \cap U_1 \rightarrow G$ for Σ ; sections of $H^0(\Sigma, P_{\mathfrak{g}^*}(p_0))$ can be represented as functions $\phi^i : U_i \rightarrow \mathfrak{g}^*$, with ϕ^1 holomorphic on U_1 and ϕ^0 meromorphic on U_0 with only one simple pole at the origin, and $\phi^1 = ad^*(T)\phi^0$ on $U_0 \cap U_1$.

We would like to split the sequence (3.1). Represent a one parameter family of elements $(P_G(t), tr(t), \phi(t))$ of $T^*\mathcal{M}_G$ by $(T(t), \phi^0(t), \phi^1(t))$, with $\phi^1(t) = Ad^*(T(t))\phi^0(t)$. At $t = 0$, the corresponding tangent vectors are given by $\dot{v} = T^{-1}\dot{T}, \dot{\phi}^0, \dot{\phi}^1$, with

$$\dot{\phi}^1 = Ad^*(T)[(ad^*(\dot{v})(\phi^0)) + \dot{\phi}^0].$$

Let

$$<, >: H^0(\Sigma, P_{\mathfrak{g}^*}(p_0)) \times H^1(\Sigma, P_{\mathfrak{g}}(-p_0)) \rightarrow \mathbb{C}$$

denote the Serre duality pairing; explicitly, it is defined by

$$<\dot{\phi}, \dot{v}> = res_{p_0}(\dot{\phi}^0 \cdot \dot{v})$$

At a point P_G of \mathcal{M}_G , choose a transition function $T = T_{10} : U_0 \cap U_1 \rightarrow G$. Let us choose a vector space V of cocycles mapping isomorphically to $H^1(\Sigma, P_{\mathfrak{g}}(-p_0))$. One can split (3.1) by taking for each \dot{v} in $V \simeq H^1(\Sigma, P_{\mathfrak{g}}(-p_0))$ the vector $(\dot{v}, \dot{\phi}_{norm})$ such that the pairing of $\dot{\phi}_{norm}$ with elements of V is zero. More generally, for any section a of $P_{\mathfrak{g}^*}(p_0)$ over U_0 , let $a^{\&}$ denote the element of $H^0(\Sigma, P_{\mathfrak{g}^*}(p_0))$ whose pairing with elements of V is the same as that of a . One then has the isomorphism

$$\begin{aligned} T(T^*\mathcal{M}_G) &\rightarrow H^1(\Sigma, P_{\mathfrak{g}}(-p_0)) \oplus H^0(\Sigma, P_{\mathfrak{g}^*}(p_0)) \\ (\dot{v}, \dot{\phi}) &\mapsto (\dot{v}, (\dot{\phi})^{\&}). \end{aligned} \quad (3.2)$$

PROPOSITION (3.3) *Under this isomorphism, the symplectic form on $T^*\mathcal{M}_G$ becomes, at (P_G, tr, ϕ) :*

$$\Omega((v', \phi'^{\&}), (\dot{v}, \dot{\phi}^{\&})) = \langle v', \dot{\phi}^{\&} \rangle - (\dot{v}, \phi'^{\&} \rangle + \langle [v', \dot{v}], \phi \rangle. \quad (3.4)$$

PROOF: One can parametrise \mathcal{M}_G locally by V ; indeed, if T is a transition matrix for P , one has in a neighbourhood of the origin a map $V \rightarrow \mathcal{M}_G$ obtained by associating to the cocycle v the transition matrix $T \cdot \exp(v)$; this in turn defines a map $\rho : V \times V^* = T^*V \rightarrow T^*\mathcal{M}_G$, which preserves the symplectic form. With respect to the splitting (3.2), the differential $d\rho$ at the origin is

$$d\rho(\dot{v}, \dot{\phi}) \rightarrow (\dot{v}, \dot{\phi} + \frac{1}{2}(ad_i^*\phi)^{\&}). \quad (3.5)$$

Substituting in the standard expression for the symplectic form on a cotangent bundle gives (3.4). ■

The explicit action of the group G on $T^*(\mathcal{M}_G)$ is given by

$$g(T, \phi^0, \phi^1) = (Ad_g(T), Ad_{g^{-1}}^*(\phi^0), \phi^1),$$

and the moment map for this action is $res_{p_0}(\phi^0)$.

From (3.4), we can compute the Hamiltonian vector fields associated to F_ω :

$$(\dot{v}, \dot{\phi}) = (df \cdot \omega, 0). \quad (3.6)$$

In other words, the Higgs field part stays as is, but the bundle varies. That this is possible is due to the invariance of the function: $ad^*(df)(\phi) = 0$. We can write an equivalent version of the flow by modifying the transition function by a coboundary: that is, we are allowed to modify our trivialisations of the bundle over the U_i , as long as on U_0 the trivialisation is not changed over p_0 . Thus, if $g_i \in H^0(U_i, P_{\mathfrak{g}}^*)$, with $g_0(0) = 0$, we have the equivalent version of the flow:

$$(\dot{v}, \dot{\phi}^0, \dot{\phi}^1) = (df \cdot \omega + Ad(T^{-1})(g_1) - g_0, ad^*(g_0)\phi^0, ad^*(g_1)\phi^1). \quad (3.7)$$

Similarly, on the reduced space, one can modify the flow by a coboundary, but now with $g_0(0)$ arbitrary.

4. A group associated to a root system

In this section, we will define the structure group which we will use for our Calogero-Moser systems. It will be associated to the root system \mathcal{R} acting on the Lie algebra \mathfrak{h} of torus $H = (\mathbb{C}^*)^r$.

We begin however with a discussion which shows that in some sense, passing to a new group is necessary. The symplectic reduction leading to the generalized Hitchin system $(T^*\mathcal{M}_G)_{\text{red}}$ depends on a choice of a coadjoint orbit of G . In our case of an elliptic base

curve, the dimension of $(T^*\mathcal{M}_G)_{\text{red}}$ is equal to the dimension of the coadjoint orbit. We are thus looking for a group, related to the rank r root system, and admitting a $2r$ -dimensional coadjoint orbit; $2r$ being the dimension of the Calogero-Moser system. When the semi-simple group is SL_n , the coadjoint orbit of $\text{diag}(1, 1, \dots, 1, 1 - n)$ is $2n - 2$ dimensional. A general semi-simple group G_{ss} does not have a $2r$ -dimensional coadjoint orbit. There is however a group G_0 , naturally associated to G_{ss} , which does admit $2r$ -dimensional coadjoint orbits. We consider the group $\mathcal{O}_0(G_{ss})$ of germs at the origin of maps $\mathbb{C} \rightarrow G_{ss}$, and let V be the subgroup of germs of the form $g(z) = h + zg_1 + z^2g_2 + \dots$, $h \in H$, and V' be the subgroup of V of germs of the form $g(z) = \text{Id} + zh_1 + z^2g_2 + \dots$, $h_1 \in \mathfrak{h}$. Then $G_0 = V/V'$. G_0 is the semi-direct product of H with the direct sum of the root spaces. The coadjoint orbits of G_0 are analyzed in Proposition (4.14).

There is a natural extension $N(G_0)$ of the Weyl group W by G_0 . Simply consider germs with a leading coefficient in the normalizer N of H in G_{ss} . $N(G_0)$ acts on \mathfrak{g}_0^* via the coadjoint action. We encounter another difficulty: For a general semi-simple group, there does not exist any W -invariant $2r$ -dimensional coadjoint orbit in \mathfrak{g}_0^* (i.e., one which is also an $N(G_0)$ orbit). Proposition (4.14) shows that for such an orbit to exist, we need a non-trivial W -invariant H -orbit in the direct sum of the root spaces. When the group G is SL_n , such an orbit is obtained by intersecting the direct sum of the root spaces with the coadjoint orbit of $\text{diag}(1, 1, \dots, 1, 1 - n)$. More generally, we relate the existence of W -invariant H -orbits to splittings of the short exact sequence

$$0 \rightarrow H \rightarrow N \rightarrow W \rightarrow 0, \quad (4.1)$$

where N is any extension of W by H such that the conjugation in N induces on H the standard W -module structure. Let V be an N representation and R a non-zero W -invariant H -orbit in V . Given a character α of H , denote by N_α and W_α its stabilizers in N and W .

LEMMA (4.2)

1) The stabilizer $\text{Stab}(\xi)$ of every element $\xi \in R$ intersects H in a fixed normal subgroup $\text{Stab}_0 \subset H$.

2) Let

$$0 \rightarrow \overline{H} \rightarrow \overline{N} \rightarrow W \rightarrow 0 \quad (4.3)$$

be the quotient of (4.1) by Stab_0 . Then the stabilizer $\text{Stab}_{\overline{N}}(\xi)$ of every $\xi \in R$ projects isomorphically onto W . In particular, (4.3) splits.

3) If V is an irreducible representation of N , and the W -invariant H -orbit R is not the zero orbit, and $\iota_\xi : W \hookrightarrow \overline{N}$ is the splitting provided by $\xi \in R$, then the representation $\iota_\xi^{-1}(V)$ of W is equivalent to $\text{Ind}_{W_\alpha}^W(1)$ for any character α of H in V . Consequently, we obtain a characterization of V as a representation of N : V is equivalent to the pullback to N of $\text{Ind}_{\overline{N}_\alpha}^{\overline{N}} \alpha$ where α is the unique character of \overline{N}_α which restricts to the trivial character of W_α and to the character α of \overline{H} .

PROOF: 1) We have the equality $\text{Stab}(n \cdot \xi) = n \cdot \text{Stab}(\xi) \cdot n^{-1}$. Since R is a H -orbit, every two stabilizers of elements in R are conjugate by an element of H . $\text{Stab}(\xi) \cap H$ is a fixed

group $Stab_0$ as H is commutative. $Stab_0$ is a normal subgroup of N since

$$[n \cdot Stab_0 \cdot n^{-1} = n \cdot Stab(\xi) \cdot n^{-1} \cap H = Stab(n \cdot \xi) \cap H = Stab_0.]$$

2) Let n_w be an element of N mapping to $w \in W$. Choose an element $\xi \in R$. Since R is also an N -orbit, there exists an element $a \in H$ such that $n_w \cdot \xi = a \cdot \xi$. Thus, $Stab(\xi)$ maps onto W . It follows that the stabilizer $Stab_{\overline{N}}(\xi)$ in \overline{N} surjects onto W . The homomorphism $Stab_{\overline{N}}(\xi) \rightarrow W$ is injective because $Stab_{\overline{N}}(\xi) \cap \overline{H} = (1)$.

3) Let α be any character of H with positive multiplicity in V , V_α the corresponding subspace, and ξ an element of R . Since V is irreducible, $\xi_\alpha \neq 0$. Since $\iota_\xi(W)$ is the stabilizer of ξ , the line spanned by ξ_α in V_α is the trivial character of W_α . It follows that the direct sum V' of all the translates of $span\{\xi_\alpha\}$ by $\iota_\xi(W)$ is a sub-representation of V . The irreducibility of V implies that $V_\alpha = span\{\xi_\alpha\}$ and $V' = V$. It follows that, as an N -module, V is the induced representation $Ind_{\overline{N}_\alpha}^{\overline{N}} V_\alpha$. The equivalence $V \cong Ind_{W_\alpha}^W(1)$ of W -modules follows. Note that V need not be irreducible as a W -representation. ■

The Lemma specifies two obstructions to the existence of a non-trivial W -invariant H -orbit in \mathfrak{g}^* for a simple Lie algebra. The first obstruction is the extension class of (4.1). In particular, considering the (co)-adjoint representation, the list of simple groups of adjoint type for which the exact sequence (4.1) for the normaliser does not split is: $SP(n)$, F_4 , E_6 , E_7 , E_8 (mod centers). The second obstruction, condition (3) in the Lemma, rules out the existence of a W -invariant H -orbit in \mathfrak{g}^* for Lie algebras of type D_n (and in the long root representation of type B_n) even though (4.1) splits.

EXAMPLE: The exact sequence (4.1) splits for $SO(2n)$ and W embeds in N as a subgroup of the group of permutation matrices. Identify the Lie algebra

$$\mathfrak{so}(2n) = \left\{ \begin{pmatrix} m & n \\ p & q \end{pmatrix} : q = -m^t, n^t = -n, \text{ and } p^t = -p \right\},$$

consider the Cartan $span\{e_{i,i} - e_{n+i,n+i}\}$ and let α be the root $\epsilon_i - \epsilon_j$ corresponding to $\mathfrak{g}_\alpha = span\{e_{i,j} - e_{n+j,n+i}\}$, $i \neq j$. The matrix of the permutation $\sigma := (i, n+j)(j, n+i)$ belongs to the stabilizer W_α of the point α in the root lattice. However, σ acts by multiplication by -1 on \mathfrak{g}_α . In particular, \mathfrak{g}_α is not the trivial character of W_α and condition (3) of the Lemma is not satisfied.

In order to circumvent the first obstruction, instead of the normaliser $N(H)$ of the torus, we will consider the semi-direct product N' of the torus and the Weyl group:

$$0 \rightarrow H \rightarrow N' \rightarrow W \rightarrow 0. \quad (4.4)$$

We now define our structure group. For any set of weights w which is invariant under the Weyl group, part (3) of Lemma (4.2) determines a representation of N' on the associated sum of weight spaces $V = \oplus \mathbb{C}_w$: indeed, if we choose a basis element for each weight space \mathbb{C}_w , the Weyl group acts simply by permuting these basis elements, while the

torus acts in the natural way on each weight space. This holds in particular for the roots $\alpha : \mathfrak{h} \rightarrow \mathbb{C}$. We define G to be the semi direct product

$$\oplus_{\alpha=1}^n \mathbb{C}_{\alpha} \rightarrow G \rightarrow N'. \quad (4.5)$$

The connected component of the identity is the group G_0 discussed above. G_0 is the semi-direct product

$$\oplus_{\alpha=1}^n \mathbb{C}_{\alpha} \rightarrow G_0 \rightarrow H. \quad (4.6)$$

Given any element of the Lie algebra \mathfrak{g} , we can decompose it into its torus and root space components; write this decomposition as $\xi = \xi_{\mathfrak{h}} + \xi_r$. Similarly, we can decompose an element a of \mathfrak{g}^* as $a_{\mathfrak{h}} + a_r$. The choice of the group G is motivated by the following:

PROPOSITION (4.7): *The $G - Ad^*$ -invariant functions on \mathfrak{g}^* only depend on the root space components, and correspond to the N' -invariant functions on $\sum_{\alpha} \mathbb{C}_{\alpha}$. The generic coadjoint orbit is $2r$ -dimensional, where $r = \dim(N')$, and is of the form*

$$(N' - \text{orbit in } \sum_{\alpha} \mathbb{C}_{\alpha}) \times \mathfrak{h}^*.$$

Moreover, \mathfrak{g}^* has a $2r$ -dimensional connected (W -invariant) coadjoint orbit.

Proposition (4.7) follows from Proposition (4.14). The rest of this section is dedicated to the proof of these two Propositions.

Let us fix a basis element for each root space \mathbb{C}_{α} , in a W -invariant way. The components of a vector $C \in \oplus_{\alpha=1}^n \mathbb{C}_{\alpha}$ are then well defined, and naturally indexed by the roots themselves: $C = (C_{\alpha})$. Let $\tilde{\alpha} : H \rightarrow \mathbb{C}^*$ denote the character corresponding to α , so that H acts on \mathbb{C}_{α} by $(h, v) \mapsto \tilde{\alpha}(h) \cdot v$. Let $C^t \cdot D$ denote the natural scalar product of two vectors in $\oplus_{\alpha=1}^n \mathbb{C}_{\alpha}$, and let $C \circ D$ denote the componentwise product: $(C \circ D)_{\alpha} = C_{\alpha} D_{\alpha}$. We denote by I the permutation matrix acting on the root spaces which permutes the α -th and the $-\alpha$ -th root spaces. Finally, we write the action of \mathfrak{h} on $\oplus_{\alpha=1}^n \mathbb{C}_{\alpha}$ as a matrix:

$$\sum_i \tau_i A_{i,\alpha} = \alpha(\tau),$$

so that the action of τ on C can be written as $(\tau^t \cdot A) \circ C$.

As a manifold, $G_0 = (\oplus_{\alpha=1}^n \mathbb{C}_{\alpha}) \times H$. The product is given by

$$(C, h)(C', h') = (C + (\exp(\log(h) \cdot A)) \circ C', hh'). \quad (4.8)$$

The corresponding Lie Bracket on $(\oplus_{\alpha=1}^n \mathbb{C}_{\alpha}) \oplus \mathfrak{h}$ is

$$[(\Gamma, \tau), (\Gamma', \tau')] = ((\tau^t \cdot A) \circ \Gamma' - (\tau'^t \cdot A) \circ \Gamma, 0). \quad (4.9)$$

There is a pairing on the Lie algebra, identifying \mathfrak{g} with \mathfrak{g}^* :

$$\langle (\Gamma, \tau), (\Gamma', \tau') \rangle = \Gamma^t \cdot I \cdot \Gamma' + \tau^t \cdot \tau'. \quad (4.10)$$

We will use this pairing to describe the coadjoint action: one then has

$$\begin{aligned} \langle [(\Delta, \sigma), (\Gamma, \tau)], (\Gamma', \tau') \rangle &= (\sigma^t \cdot A \circ \Gamma - \tau^t \cdot A \circ \Delta)^t \cdot I \cdot \Gamma' \\ &= -\Gamma^t \cdot I \cdot ((\sigma^t \cdot A) \circ \Gamma') - \tau^t \cdot A \cdot (\Delta \circ (I \cdot \Gamma')), \end{aligned} \quad (4.11)$$

remembering that $A_{i,-\alpha} = -A_{i,\alpha}$. Therefore

$$ad_{(\Delta, \sigma)}^*(\Gamma', \tau') = ((-\sigma^t \cdot A) \circ \Gamma', -A \cdot (\Delta \circ (I \cdot \Gamma'))). \quad (4.12)$$

Similarly, the coadjoint action of an element of the group can be written as

$$Ad_{(D, \exp(\sigma))}^*(\Gamma', \tau') = (\exp(-\sigma^t \cdot A) \circ \Gamma', \tau' - A \cdot (D \circ (I \cdot \Gamma'))). \quad (4.13)$$

From (4.13), one has:

PROPOSITION (4.14): *The $G_0 - Ad^*$ -invariant functions on \mathfrak{g}^* only depend on the root space components, and correspond to the H -invariant functions on $\sum_{\alpha} \mathbb{C}_{\alpha}$. The generic coadjoint orbit is $2r$ -dimensional, where $r = \dim(H)$, and is of the form $(H\text{-orbit in } \sum_{\alpha} \mathbb{C}_{\alpha}) \times \mathfrak{h}^*$.*

The pairing (4.10) is part of a more general family of invariant inner products on \mathfrak{g} :

LEMMA (4.15) *Let D be a diagonal matrix such that $D_{\alpha, \alpha} = D_{\alpha', \alpha'}$ if α, α' lie in the same Weyl group orbit, and let δ be a constant. The N' invariant pairings on \mathfrak{g} are given by*

$$\langle (\Gamma, \tau), ((\Gamma', \tau')) \rangle_{D, \delta} = \Gamma^t \cdot D \cdot I \cdot \Gamma' + \delta \tau^t \cdot \tau'. \quad (4.16)$$

When there is a single Weyl orbit of roots, there is then a two parameter family of pairings; when there two orbits, there is a three parameter family.

PROOF: The invariance under H forces the α -th root space to be paired only with the $-\alpha$ -th, and \mathfrak{h} to be paired only with itself. Further invariance under the Weyl group reduces one to a single choice up to scale for \mathfrak{h} and for each Weyl orbit of roots.

5. Hitchin systems for G

We now turn to studying (modified) Hitchin systems for our group G , over an elliptic curve Σ . We begin with

$$\mathcal{M}_G = \{G\text{-bundles of degree zero, trivialized at } p_0\},$$

then take the cotangent bundle of this space. We then reduce by the action of the group G , at a W -invariant element of \mathfrak{g}^* ; this element must then lie in \mathfrak{g}_r^* . We will see that this is essentially equivalent to reducing by the action of the subgroup $\oplus_{\alpha} \mathbb{C}_{\alpha}$. The Calogero-Moser Hamiltonians are then expressed naturally in terms of the scalar product of (4.10) on the reduced space.

5.i. G -bundles on Σ

We begin by giving an explicit description of the moduli of framed G -bundles of degree 0 on Σ . We first note that under the projection of G to W , any G -bundle defines a W -bundle. We will consider only the component of moduli corresponding to trivial W -bundles, so that we can represent our bundles as G_0 -bundles. The subspace of G -bundles which can be represented as G_0 bundles is a quotient of the moduli of G_0 -bundles: one must quotient out by the action of W , since different G_0 -bundles can be the same as G -bundles.

$$\mathcal{M}_G = \mathcal{M}_{G_0}/W. \quad (5.1)$$

Let $\overline{\mathcal{M}}_{G_0}$ be the moduli of G_0 -bundles (without framing). To analyse the moduli $\overline{\mathcal{M}}_{G_0}$, we use the fact that the group G_0 maps to H , and so one has maps

$$\Pi : \overline{\mathcal{M}}_{G_0} \rightarrow \overline{\mathcal{M}}_H = \text{Pic}^0(\Sigma)^r = \Sigma^r, \quad (5.2)$$

$$\Pi_W : \overline{\mathcal{M}}_{G_0}/W \rightarrow \overline{\mathcal{M}}_H/W = \Sigma^r/W. \quad (5.3)$$

By a theorem of Looijenga [Lo], the space Σ^r/W is a weighted projective space. The fiber of (5.1) at $\chi \in \Sigma^r$ is $\oplus_\alpha H^1(\Sigma, L_{\tilde{\alpha}(\chi)})$, where $L_{\tilde{\alpha}(\chi)}$ is the line bundle associated to $\tilde{\alpha}(\chi)$. This can be seen by writing out a cocycle explicitly in the semi-direct product. Each $H^1(\Sigma, L_{\tilde{\alpha}(\chi)})$ is isomorphic to \mathbb{C} if $L_{\tilde{\alpha}(\chi)}$ is trivial and is (0) otherwise. Consequently, one has an open set $\overline{\mathcal{M}}'_{G_0} \subset \overline{\mathcal{M}}_{G_0}$ isomorphic to the open set of Σ^r/W corresponding to H -bundles for which none of the $L_{\tilde{\alpha}(T_{\mathfrak{h}})}$ are trivial.

Putting the framings back in, one has that the moduli of framed H -bundles is the same as the moduli of unframed H -bundles, as the automorphisms act transitively on framings. Consequently, one has

$$\Pi : \mathcal{M}_{G_0} \rightarrow \overline{\mathcal{M}}_H = \text{Pic}^0(\Sigma)^r = \Sigma^r, \quad (5.4)$$

$$\Pi_W : \mathcal{M}_{G_0}/W \rightarrow \overline{\mathcal{M}}_H/W = \Sigma^r/W. \quad (5.5).$$

This time, the fibre is $\oplus_\alpha H^1(\Sigma, L_{\tilde{\alpha}(\chi)}(-p_0))$. Each $H^1(\Sigma, L_{\tilde{\alpha}(\chi)}(-p_0))$ is isomorphic to \mathbb{C} .

DEFINITION: We say that a G -bundle P_G is special if $L_{\tilde{\alpha}(T_{\mathfrak{h}})}$ is trivial for some root α .

One has an open set

$$\mathcal{M}'_G \subset \mathcal{M}_G$$

of framed non-special G -bundles. We will take a reduction of \mathcal{M}'_G , which will be the space over which the Calogero-Moser systems are defined, and indeed, we shall see that the reduction does not extend to the locus where one of the $L_{\tilde{\alpha}(T_{\mathfrak{h}})}$ is trivial.

Explicitly, covering the elliptic curve by $U_0 = \text{disk around } p_0$, and $U_1 = \Sigma - p_0$, the torus part $T_{\mathfrak{h}}$ of the transition function T is a function $T_{\mathfrak{h}} : U_0 \cap U_1 \rightarrow H$. We can choose these functions to be of the form

$$T_{\mathfrak{h}} = \exp(x\zeta(z)), \quad x \in \mathfrak{h}. \quad (5.6)$$

The root space part T_r can be represented by a vector M of cocycles M_α representing elements of $H^1(\Sigma, L_{\tilde{\alpha}(T_{\mathfrak{h}})}(p_0))$. For $L_{\tilde{\alpha}(T_{\mathfrak{h}})}$ non trivial, these cocycles can be taken to be constant functions on $U_0 \cap U_1$; when $L_{\tilde{\alpha}(T_{\mathfrak{h}})}$ is trivial, the constant functions correspond to trivial classes, and one must choose a function with a simple pole at p_0 as generator, for example $\zeta(z)$.

One has as cotangent space to \mathcal{M}'_G at a bundle P_G the set of Higgs fields ϕ in $H^0(\Sigma, P_{\mathfrak{g}^*}(p_0))$ (fields in the associated coadjoint bundle with poles at p_0). Splitting ϕ into a root space component and a torus component,

$$\phi = \phi_{\mathfrak{h}} + \phi_r, \quad (5.7)$$

one has that the components ϕ_α of ϕ_r have poles at the origin only when the line bundle is not trivial. Explicitly, for a bundle with transition functions $(T_r, T_{\mathfrak{h}} = \exp(x\zeta(z)))$, one represents $(\phi_r, \phi_{\mathfrak{h}})$ in the U_0 -trivialisation by $(\phi_r^0, \phi_{\mathfrak{h}}^0)$, and in the U_1 -trivialisation by $(\phi_r^1, \phi_{\mathfrak{h}}^1)$, with

$$(\phi_r^1, \phi_{\mathfrak{h}}^1) = (\exp(-x^t \cdot A\zeta(z)) \circ \phi_r^0, \phi_{\mathfrak{h}}^0 - A \cdot (T_r \circ (I \cdot \phi_r^0))).$$

Here ϕ_r^0 has simple poles at the origin; its components ϕ_α^0 are simply multiples of the functions $\rho^0(\alpha(x), z)$ of (2.4). Similarly, the components ϕ_α^1 of ϕ_r^1 are multiples of the functions $\rho^1(\alpha(x), z)$.

5.ii. Reduction

There is, as in section 3, an action of G on $T^*\mathcal{M}_G$ by changing the trivialisation at p_0 . The moment map for this action is, as we saw in section 3, simply the residue of the Higgs field ϕ at p_0 , expressed in the trivialization. We want to reduce at an element C of \mathfrak{g}^* which is W invariant. Being W -invariant, C must lie in \mathfrak{g}_r^* . W -invariance implies that reduction of $T^*\mathcal{M}_G$ by G at C is equivalent to the reduction of $T^*\mathcal{M}_{G_0}$ by G_0 at C , then quotienting by W . The set of root vectors α splits up into W -orbits according to their lengths $|\alpha|$, and we choose constants $c_{|\alpha|}$ for each length, and set $C = (c_{|\alpha_1|}, c_{|\alpha_2|}, \dots, c_{|\alpha_n|})$. Note that the coadjoint orbit $G_0 \cdot C$ is also a G -orbit, the connected orbit of Proposition (4.7).

If the constants $c_{|\alpha_i|}$ are non-zero, the coadjoint orbit of the element C is of the form $(H\text{-orbit in } \mathfrak{g}_r^*) \times \mathfrak{h}^*$. Also, the stabiliser of the element C lies in the root space part of the group. From these facts and from the expressions for the actions, it follows that taking the symplectic quotient by G_0 or the quotient by its subgroup $\oplus_\alpha \mathbb{C}_\alpha$ gives exactly the same result.

The action of an element $V \in \oplus_\alpha \mathbb{C}_\alpha$ on $(T, \phi^0) \in T^*\mathcal{M}_G$ is given explicitly by

$$(T, \phi^0) = ((T_r, T_{\mathfrak{h}}), (\phi_r^0, \phi_{\mathfrak{h}}^0)) \mapsto ((T_r + V, T_{\mathfrak{h}}), (\phi_r^0, \phi_{\mathfrak{h}}^0 - A \cdot (V \circ (I \cdot \phi_r^0))). \quad (5.8)$$

The moment map for this action is $\text{res}(\phi_r^0)$.

Reducing at C , we fix to C the residues of ϕ_r^0 , and quotient by the group. Referring to the explicit form of the action, this means that we can normalise to $T_r = 0$, thus reducing the bundle to the torus, over the locus \mathcal{M}'_G . Our reduced space $(T^*\mathcal{M}_G)_{\text{red}}$ can thus be thought of as a subspace of the unreduced one. This subspace $(T^*\mathcal{M}_G)_{\text{red}}$ of \mathcal{M}_G is characterised by:

$$T_r = 0, \quad \text{res}(\phi_r^0) = \text{a } W - \text{invariant constant } C. \quad (5.9)$$

Note that we still have the torus part of the framing in this description of our reduced space.

Explicitly, the elements of $(T^*\mathcal{M}_G)_{\text{red}}$ are then H -bundles with transition functions

$$T = (T_r, T_{\mathfrak{h}}) = (0, \exp(x\zeta(z))),$$

along with Higgs fields whose root space components are, in the U_0 -trivialisation

$$\phi_\alpha^0 = c_{|\alpha|} \rho^0(\alpha(x), z),$$

and whose torus components are constants

$$\phi_{\mathfrak{h}}^0 = \phi_{\mathfrak{h}}^1 = p.$$

By (3.3) the functions $x \in \mathfrak{h}, p \in \mathfrak{h}^*$ provide canonical coordinates on $(T^*\mathcal{M}_G)_{\text{red}}$; one must remember that we are restricting to the set $\alpha(x) \neq 0, \alpha \in \mathcal{R}$, and quotienting out by the action of the affine Weyl group on $\mathfrak{h} \times \mathfrak{h}^*$.

Remarks: 1) In the above discussion we have excluded a Higgs pair (P_G, ϕ) if the bundle P_G is special (see section 5.i). This exclusion is *automatically imposed* by the reduction along the coadjoint orbit $G \cdot C$. The coadjoint bundle $P_{\mathfrak{g}^*}$ fits in the short exact sequence

$$0 \rightarrow \mathfrak{h}^* \rightarrow P_{\mathfrak{g}^*} \rightarrow (\oplus_{\tilde{\alpha}} L_{\tilde{\alpha}})^* \rightarrow 0.$$

The moment map sends (P_G, ϕ) into $G \cdot C$ if and only if the residue of ϕ projects into the H -orbit of C in the fiber of $(\oplus_{\tilde{\alpha}} L_{\tilde{\alpha}})^*(p_0)$ at p_0 . The triviality of one of the $L_{\tilde{\alpha}}$ rules out the existence of such sections in $H^0(\Sigma, P_{\mathfrak{g}^*}(p_0))$.

2) All the non-special principal G -bundles considered have a canonical reduction to a principal N' -bundle (equivalently, a W -orbit of reductions to an H -bundle). This follows from the fact that the global sections of $P_{\mathfrak{g}}$ generate a sub-bundle of commutative sub-algebras isomorphic to \mathfrak{h} .

5.iii The Calogero-Moser Hamiltonian

Let ω be the class in $H^1(\Sigma, \mathcal{O})$, with representative with respect to the cover U_0, U_1

$$\omega = \zeta(z). \quad (5.10)$$

Our Hamiltonian on $(T^*\mathcal{M}_G)_{\text{red}}$ will then be the W-invariant function on the reduced space:

$$CM = \text{res}_{p_0}(\omega < \phi, \phi >)\}. \quad (5.11)$$

Note that the bilinear form of Lemma 4.15 gives rise to a canonical bilinear form on $P_{\mathfrak{g}}$ because all the non-special principal G -bundles we consider have a canonical reduction to N' . The bilinear form sends a Higgs pair (P_G, ϕ) to the element $< \phi, \phi >$ of $H^0(\Sigma, K_{\Sigma}^{\otimes 2}(2p_0))$. $H^0(\Sigma, K_{\Sigma}^{\otimes 2}(2p_0))$ is two dimensional. If the pair (P_G, ϕ) belongs to $(T^*\mathcal{M}_G)_{\text{red}}$, then the residue of ϕ belongs to our coadjoint orbit $G \cdot C$. Hence, the quadratic residue of $< \phi, \phi >$ is fixed and $< \phi, \phi >$ lies in a marked affine line ℓ in $H^0(\Sigma, K_{\Sigma}^{\otimes 2}(2p_0))$. We see that the Calogero-Moser Hamiltonian is determined canonically up to a choice of an affine linear isomorphism $\ell \cong \mathbb{C}$. The restriction of the linear functional (5.11) on $H^0(\Sigma, K_{\Sigma}^{\otimes 2}(2p_0))$ provides such an isomorphism.

One can split CM into a sum $CM_r + CM_{\mathfrak{h}}$ of a root space piece $CM_r = \{\omega, < \phi_r, \phi_r >\} = \{\omega, \sum \phi_{\alpha}^0 \phi_{-\alpha}^0\}$ and a torus piece $CM_{\mathfrak{h}} = \{\omega, < \phi_{\mathfrak{h}}, \phi_{\mathfrak{h}} >\}$. The relation (2.8) tells us that

$$\phi_{\alpha}^0 \phi_{-\alpha}^0 = c_{|\alpha|}^2 (\mathfrak{p}(z) - \mathfrak{p}(\alpha(x)))$$

and so the Hamiltonian is

$$CM = p \cdot p - \sum_{\alpha} c_{|\alpha|}^2 \mathfrak{p}(\alpha(x)), \quad (5.12)$$

which is indeed the Calogero-Moser Hamiltonian, setting $m_{|\alpha|} = -c_{|\alpha|}^2$.

Next we provide an explicit formula for the vector field of the Calogero-Moser Hamiltonian. The formula (5.18) will be needed in Section 6. The function CM_r is ad^* -invariant, and its differential on the $H^0(\Sigma, P_{\mathfrak{g}^*}(p_0))$ -component of the tangent space $T^*\mathcal{M}_G$ is then

$$dCM_r = (\omega\phi)_r^0, \quad (5.13)$$

where one thinks of dCM_r as an element of $H^1(\Sigma, P_{\mathfrak{g}}(-p_0))$ acting on $H^0(\Sigma, P_{\mathfrak{g}^*}(p_0))$. With respect to the splitting (3.2) the action of dCM_r on the $H^1(\Sigma, P_{\mathfrak{g}}(-p_0))$ -component of the tangent space is trivial. By the considerations of section 2, the Hamiltonian vector field in $T^*\mathcal{M}_G$ of CM_r at (T, ϕ^0, ϕ^1) , where T is the (torus) transition matrix and $\phi^1 = Ad^*(T)(\phi^0)$ is the Higgs field, is given by:

$$T^{-1}\dot{T} = \omega\phi_r^0, \quad \dot{\phi}^i = 0 \quad (5.14)$$

This takes us out of the normalised form for the reduced space, since the transition function no longer lies in the torus. Remembering that we are on an elliptic curve, the root space components $\omega\phi_{\alpha}$ in $H^1(\Sigma, L_{\alpha})$ (in the reduced space, we have forgotten the root space component of the framings, and so we are in $H^1(\Sigma, L_{\alpha})$ instead of $H^1(\Sigma, L_{\alpha})(-p_0)$) can be written as coboundaries:

$$(\omega\phi)_r = (\omega\phi)_r^0 + Ad_{T^{-1}}(\omega\phi)_r^1 \quad (5.15)$$

and the flow of CM_r can be written

$$\begin{aligned} T^{-1}\dot{T} &= 0, \\ \dot{\phi}^0 &= (ad_{(\omega\phi)_r^0}^*\phi^0)^{\&}. \end{aligned} \quad (5.16)$$

The vector field on $(T^*\mathcal{M}_G)_{\text{red}}$ corresponding to $CM_{\mathfrak{h}} = p \cdot p$, with respect to the splitting (3.2) is simply

$$\begin{aligned} T^{-1}\dot{T} &= \omega\phi_{\mathfrak{h}}^0 = \omega p, \\ \dot{\phi}^0 &= 0 \end{aligned} \quad (5.17)$$

and so combining (5.16) and (5.17), one has for the flow of CM

$$\begin{aligned} T^{-1}\dot{T} &= \omega\phi_{\mathfrak{h}}^0, \\ \dot{\phi}^0 &= (ad_{(\omega\phi_r^0)}^*\phi^0)^{\&}. \end{aligned} \quad (5.18)$$

Let us check that the vector field (5.18) is indeed the vector field that one obtains from the explicit parametrisation. From (2.6), using $\omega = \zeta(z)$ we find that the α -th components of the coboundary decomposition (5.15) satisfy

$$(\omega\phi)_{\alpha}^0 A_{i,\alpha} = \frac{d}{dx_i}(\phi^0)_{\alpha} = c_{|\alpha|} \frac{d}{dx_i} \rho^0(\alpha(x), z), \quad (5.19)$$

where we decompose $x \in \mathfrak{h}$ into components x_1, \dots, x_r , and $\alpha_i = \frac{d\alpha}{dx_i}$ is the corresponding component of the root α . Referring to the formulae (4.12) for the coadjoint action and to (2.8), we have for the flows:

$$\begin{aligned} \dot{p}_i &= \sum_{\alpha} (c_{|\alpha|} \frac{d}{dx_i} \rho^0(\alpha(x), z)) \cdot c_{|\alpha|} \rho^0(-\alpha(x), z) \\ &= \frac{1}{2} \sum_{\alpha} (c_{|\alpha|}^2 \frac{d}{dx_i} \mathfrak{p}(\alpha(x))), \\ \dot{x}_i &= p_i. \end{aligned} \quad (5.20)$$

6. Embeddings in $Gl(N, \mathbb{C})$.

We now give two embeddings of our system into the Hitchin systems for $Gl(N, \mathbb{C})$ over Σ , one corresponding to the Lax pairs of [dP1], the other to those of [BCS2].

Let $V = \mathbb{C}^N$ be a sum of (integral) weight spaces $\mathbb{C}_{\omega_i}, i = 1, \dots, N$ for the torus H , such that the set of roots is Weyl invariant. The weights w_i are maps of \mathfrak{h} to \mathbb{C} ; denote the corresponding homomorphism $H \rightarrow \mathbb{C}^*$ by \tilde{w}_i . As for the root spaces, each of these weight spaces should be thought of as having a preferred basis, and the bases are invariant under the Weyl group.

One has an embedding Ξ of the torus H into the diagonal subgroup D of $gl(N, \mathbb{C})$; it is given by

$$\Xi(h) = \text{diag}(\tilde{w}_1(h), \dots, \tilde{w}_N(h)). \quad (6.1)$$

Let ξ denote the corresponding Lie algebra homomorphism.

The homomorphism Ξ induces a map $\hat{\Xi}$ from the space of H -bundles of degree 0 over Σ to the space of D -bundles of degree 0 over Σ , where D is the diagonal subgroup of $Gl(N)$. The space of H -bundles, as we saw, can be parametrised with some redundancy by \mathfrak{h} ; for $h \in \mathfrak{h}$, the corresponding D -bundle $\hat{\Xi}(E)$ is a sum of line bundles $\oplus_i L_{w_i(h)}$. The bundle $\text{End}(\hat{\Xi}(E))$ is then a sum of line bundles $\oplus_{i,j} L_{w_i(h) - w_j(h)}$. The differences $w_i - w_j$ are sums of roots, for w_i, w_j in the same orbit. We note that the space of $Gl(N, \mathbb{C})$ -bundles is essentially the finite quotient of the space of D -bundles by the Weyl group of $Gl(N)$, as the generic $Gl(N, \mathbb{C})$ -bundle reduces to the torus.

The d'Hoker-Phong embedding

The map $\hat{\Xi}$ extends to a map

$$\Xi_{\mathcal{M}} : (T^* \mathcal{M}_G)_{\text{red}} \rightarrow T^* \mathcal{M}_{GL(N)} \quad (6.2)$$

of the reduced moduli space $(T^* \mathcal{M}_G)_{\text{red}}$ into the cotangent bundle $T^* \mathcal{M}_{GL(N)}$ of the space of $Gl(N)$ -bundles with level structure at the point p_0 of Σ .

Conceptually, the map $\Xi_{\mathcal{M}}$ is determined by an N' -equivariant extension of ξ^* to a linear map from \mathfrak{g}^* to $gl(N, \mathbb{C})$. Recall that a non-special G -bundle admits a canonical reduction to a N' -bundle. Thus, $(T^* \mathcal{M}_G)_{\text{red}}$ is also a moduli space of pairs (P, ϕ) where P is a principal N' -bundle, $P_{\mathfrak{g}^*}$ is the vector bundle associated via the map of \mathfrak{g}^* into $gl(N, \mathbb{C})$, and ϕ is a section of $P_{\mathfrak{g}^*} \otimes K(p_0)$. The homomorphism Ξ extends to a homomorphism $\Xi : N' \rightarrow N(D)$ realizing $gl(N, \mathbb{C})$ as an N' representation. Thus, an N' -equivariant linear map from \mathfrak{g}^* to $gl(N, \mathbb{C})$ gives rise to a map (6.2).

More explicitly, $(T^* \mathcal{M}_G)_{\text{red}}$ is a space of pairs

(H -bundle E with H -level structure at p_0 , section ϕ_G of $E(\mathfrak{t} \oplus (\oplus_{\alpha} \mathbb{C} \alpha)) \otimes K_{\Sigma}(p_0)$).

To this, $\Xi_{\mathcal{M}}$ will associate an element of $T^* \mathcal{M}_{GL(N)}$. Such an element is a pair (rk N bundle $E_{GL(N)}$ with level structure at p_0 , section $\phi_{GL(N)}$ of $\text{End}(E_{GL(N)}) \otimes K_{\Sigma}(p_0)$).

The $Gl(N, \mathbb{C})$ -bundle $E_{GL(N)}$ associated by $\Xi_{\mathcal{M}}$ to (E, ϕ) is simply $\hat{\Xi}(E)$. We then define the corresponding $\phi_{GL(N)}$. We choose for each pair (w, w') of weights a constant $C_{w, w'}$ in a way that it is invariant under the Weyl group and so that

$$C_{w, w'} = C_{w', w}.$$

We then define a “shift” operator for each root α

$$(Sh_{\alpha})_{w, w'} = \delta_{w - w', \alpha} C_{w, w'}, \quad (6.3)$$

where we index the entries of the matrix by the weights themselves. The coefficient $\delta_{w-w',\alpha}$ is the Kronecker δ . We then set

$$\phi_{Gl(N)} = \xi((\phi_G)_{\mathfrak{h}}) + \sum_{\alpha \in \mathcal{R}} (\phi_G)_{\alpha} S h_{\alpha}. \quad (6.4)$$

Let \mathcal{CM} denote the image $\Xi((T^*\mathcal{M}_G)_{\text{red}})$. The space $\mathcal{M}_{GL(N)}$ has dimension N^2 . Indeed, the space of bundles is of dimension N : the generic $GL(N)$ -bundle on Σ reduces to the subgroup D of diagonal matrices. The bundles have, generically, the group D as automorphisms. When one adds in the level structure, one adds in N^2 parameters, on which the automorphisms act, reducing one to $N^2 - N$ parameters, giving $N + N^2 - N = N^2$ parameters in all. When one considers the Higgs fields $\phi_{Gl(N)}$ in $H^0(\Sigma, \text{End}(E_{Gl(N)})) \otimes K_{\Sigma}(p_0)$, one has similarly N^2 parameters, giving $2N^2$ parameters for $T^*\mathcal{M}_{GL(N)}$. The Calogero-Moser locus \mathcal{CM} lies inside $T^*\mathcal{M}_{GL(N)}$, and is of dimension $2r$. It is characterised by the fact that the framing is compatible with the reduction to the diagonal subgroup $\Xi(H) \subset D$ (so that transition functions respecting the trivialisation can be chosen diagonal), and the polar parts of the Higgs field $\phi_{Gl(N)}$ are fixed, while its diagonal parts lie in $\xi(\mathfrak{h})$

More explicitly, for any matrix A , let

$$A = A_d + A_{od} \quad (6.5)$$

denote the splitting of A into diagonal and off-diagonal matrices. One can choose the transition matrices M for a $GL(N)$ -bundles, at least generically, to be diagonal, so that

$$M_d = \text{diag}(\exp(y_i \zeta(z))), \quad M_{od} = 0, \quad (6.6)$$

where y_i are constant on Σ . In turn, one represents the Higgs fields, which decompose into a sum of sections of line bundles, by

$$\begin{aligned} (\phi_{Gl(N)})_d &= \text{diag}(q_i), \\ ((\phi_{Gl(N)}^0)_{od})_{w,w'} &= C_{w,w'} \left(\sum_{\alpha} \delta_{w-w',\alpha} c_{|\alpha|} \right) \rho^0(\alpha(x), z), \end{aligned} \quad (6.7)$$

where q_i are constant functions, and ρ^0 are the functions of (2.4). Note that the $K_{w,w'}$ are the residues of the section at the origin. The Calogero-Moser locus \mathcal{CM} is given by constraints

$$\begin{aligned} (1) \quad & \text{diag}(y_i) \subset \mathfrak{h}, \\ (2) \quad & M_{od} = 0, \\ (3) \quad & \text{diag}(q_i) \subset \mathfrak{h}, \\ (4) \quad & \text{res}_0((\phi_{Gl(N)}^0)_{w,w'}) = C_{w,w'} \sum_{\alpha} \delta_{w-w',\alpha} c_{|\alpha|}. \end{aligned} \quad (6.8)$$

Referring to the explicit form of the symplectic form on $T^*\mathcal{M}_{GL(N)}$ (3.3), it follows:

PROPOSITION (6.9): *The embedding $\Xi_{\mathcal{M}}$ is symplectic, and the Calogero-Moser Hamiltonian corresponds under the embedding to a multiple of $\text{res}(\omega \text{tr}(\phi_{Gl(N)}^2))$.*

We need to compare the flow in the reduced space of G -bundles and the flow in the cotangent space of $Gl(N)$ -bundles with level structure; in our $Gl(N)$ moduli, the Hamiltonian is given, up to a constant, by pairing $\text{tr}(\phi_{Gl(N)}^2)$ with the cocycle ω of (5.7); the corresponding flow of $(M, \phi_{Gl(N)})$ is, by (3.6)

$$M^{-1}\dot{M} = \omega\phi_{Gl(N)}^0, \quad \dot{\phi}_{Gl(N)}^0 = 0. \quad (6.10)$$

Again we split $\omega\phi_{Gl(N)}^0$, first into its diagonal and off-diagonal components, and then write the off diagonal term as a coboundary (function on U_0 vanishing at the origin, minus M^{-1} (function on U_1) plus a constant cocycle:

$$\omega\phi_{Gl(N)}^0 = (\omega\phi_{Gl(N)}^0)_d + (\omega\phi_{Gl(N)}^0)_{od}^0 - M^{-1}(\omega\phi_{Gl(N)}^0)_{od}^1 M + (\omega\phi_{Gl(N)}^0)_{od}^{\text{cst}} \quad (6.11)$$

referring to (3.7) this transforms the flow (6.12) into the equivalent one:

$$\begin{aligned} M^{-1}\dot{M} &= (\omega\phi_{Gl(N)}^0)_d + (\omega\phi_{Gl(N)}^0)_{od}^{\text{cst}}, \\ (\dot{\phi}_{Gl(N)}^0) &= [(\omega\phi_{Gl(N)}^0)_{od}^0, (\phi_{Gl(N)}^0)], \end{aligned} \quad (6.12)$$

This is not necessarily tangent to the embedded Calogero-Moser locus \mathcal{CM} : it does not satisfy the constraints (1), (2) and (3) of (6.7), but does satisfy the constraint (4). When one has a symplectic subvariety V of a larger subvariety W , one can split the tangent space of W along V into $TV \oplus (TV)^\perp$, using the symplectic form. The Hamiltonian vector field of H along V is simply the projection of the corresponding field in W , with respect to this splitting. Referring to the formula (3.4), splitting the diagonal matrices as $\mathfrak{d} = \mathfrak{h} \oplus \mathfrak{h}^\perp$, and letting $\pi_{\mathfrak{h}} : gl(N) \rightarrow \mathfrak{h}, \pi_{\mathfrak{h}^\perp} : gl(N) \rightarrow \mathfrak{h}^\perp$ be the ensuing projections, the bundle $(TC\mathcal{M})^\perp$ is given by

$$\begin{aligned} \pi_{\mathfrak{h}}(M^{-1}\dot{M}) &= 0 \\ \pi_{\mathfrak{h}}(\dot{\phi}_{Gl(N)}^0 + [M^{-1}\dot{M}, \phi_{Gl(N)}^0]) &= 0 \end{aligned} \quad (6.13)$$

One can make the $\mathcal{M}_{GL(N)}$ -flow tangent to the Calogero-Moser locus V by adding to it the following vector field, which lies in $TC\mathcal{M}^\perp$

$$\begin{aligned} M^{-1}\dot{M} &= -(\omega\phi_{Gl(N)}^0)_{od}^{\text{cst}} \\ (\dot{\phi}_{Gl(N)}^0) &= a + [\omega\phi_{Gl(N)}^0]_{od}^{\text{cst}}, (\phi_{Gl(N)}^0)_d \end{aligned} \quad (6.14)$$

where $a = a(x)$ is a suitable constant (in z) in \mathfrak{h}^\perp , giving the Calogero-Moser flow:

$$\begin{aligned} M^{-1}\dot{M} &= (\omega\phi_{Gl(N)}^0)_d \\ (\dot{\phi}_{Gl(N)}^0) &= a(x) + [(\omega\phi_{Gl(N)}^0)_{od}^0, (\phi_{Gl(N)}^0)] + [\omega\phi_{Gl(N)}^0]_{od}^{\text{cst}}, (\phi_{Gl(N)}^0)_d, \\ &= a(x) + [(\omega\phi_{Gl(N)}^0)_{od}^0 + (\omega\phi_{Gl(N)}^0)_{od}^{\text{cst}}, (\phi_{Gl(N)}^0)] - [(\omega\phi_{Gl(N)}^0)_{od}^{\text{cst}}, (\phi_{Gl(N)}^0)]_{od}, \end{aligned} \quad (6.15)$$

Indeed, this satisfies the constraints (1) (2) (4) of (6.8), the third constraint being given by an appropriate choice of $a \in \mathfrak{h}^\perp \subset \mathfrak{d}$:

$$a(x) = -\pi_{\mathfrak{h}^\perp}([(\omega \phi_{Gl(N)}^0)_{od}^0 + (\omega \phi_{Gl(N)}^0)_{od}^{\text{cst}}, (\phi_{Gl(N)}^0)]). \quad (6.16)$$

Let D' be the group of automorphisms of the bundle given by M ; with respect to our trivialisations the automorphisms are represented by constant matrices. D' includes the group D of diagonal matrices. The action of D' on $T^* \mathcal{M}_{GL(N)}$ is represented by the vector field:

$$M^{-1} \dot{M} = 0, \dot{\phi}_{Gl(N)}^0 = [d', \phi_{Gl(N)}^0], \quad (6.17)$$

for $d' \in \text{Lie}(D')$. If d' is diagonal, this vector field lies in \mathcal{CM}^\perp .

We would like to use this vector field to rewrite the flows (6.15) as

$$\begin{aligned} M^{-1} \dot{M} &= (\omega \phi_{Gl(N)}^0)_d, \\ (\dot{\phi}_{Gl(N)}^0) &= [(\omega \phi_{Gl(N)}^0)_{od}^0 + \omega \phi_{Gl(N)}^0]_{od}^{\text{cst}} + d'(x), (\phi_{Gl(N)}^0)], \end{aligned} \quad (6.18)$$

giving a Lax pair with spectral parameter for the flow. This gives the constraint

$$[d'(x), (\phi_{Gl(N)}^0)] = a - [(\omega \phi_{Gl(N)}^0)_{od}^{\text{cst}}, (\phi_{Gl(N)}^0)]_{od}. \quad (6.19)$$

We have, referring to (2.6)-(2.8),

$$\begin{aligned} (\phi^0)_{w, w+\alpha}(x, z) &= C_{w, w+\alpha} c_\alpha(-z^{-1} + \zeta(\alpha(x)) + O(z)), \\ &\stackrel{\text{def}}{=} R_{w, w+\alpha} z^{-1} + Q_{w, w+\alpha}(x) + O(z). \end{aligned} \quad (6.20)$$

$$\begin{aligned} ((\omega \phi_{Gl(N)}^0)_{od}^0 + \omega \phi_{Gl(N)}^0)_{w, w+\alpha} &= C_{w, w+\alpha} c_\alpha \frac{d}{d\alpha(x)} (\rho^0(\alpha(x), z)) \\ &= R_{w, w+\alpha} \mathbf{p}(\alpha(x)) + O(z) \\ &\stackrel{\text{def}}{=} P_{w, w+\alpha}(x) + O(z). \end{aligned} \quad (6.21)$$

Recall that we are all along dealing with flows of bundles and of sections $\phi_{Gl(N)}$, and in particular, that sections are determined by their leading order terms at $z = 0$. This gives necessary and sufficient algebraic constraints for $d' = d'(x)$:

$$[P(x), R]_{od} = [d'(x), R]_{od}, \quad (6.22)$$

$$0 = [d'(x), R]_d, \quad (6.23)$$

$$a(x) = [d'(x), Q(x)]_d. \quad (6.24)$$

Relation (6.23) is automatically satisfied and, when d' is diagonal, (6.24) forces $a = 0$.

These algebraic constraints are essentially the ones of Theorems 1 and 2 of [dP1]. Indeed, their theorem 1 gives an ansatz for a Lax pair, which contains our solution: they have three constraints, labelled there (3.7), (3.8), and (3.9); they then particularise their ansatz in Theorem 2 to what is in essence our case, with d' diagonal; their conditions then particularise to their (3.17), (3.18), (3.19). The first of their conditions follows automatically from Weyl invariance; their second, (3.18), essentially tells us that $a = 0$; their third is condition (6.22). By choosing a suitable representation (which is strongly constrained by the conditions), they then ensure that these conditions can be satisfied.

The flow then has the Lax form (6.18) on the unreduced space $T^*\mathcal{M}_{GL(N)}$. Projecting to the reduced space, one quotients out by the action of the automorphisms of the bundle, and so omits the $d(x)$, giving simply

$$\begin{aligned} M^{-1}\dot{M} &= (\omega\phi_{GL(N)}^0)_d, \\ (\dot{\phi}_{GL(N)}^0) &= [(\omega\phi_{GL(N)}^0)_{od}^0 + \omega\phi_{GL(N)}^0{}_{od}^{\text{cst}}, (\phi_{GL(N)}^0)], \end{aligned} \quad (6.25)$$

which is precisely the flow of the Hitchin system. In particular, one has a full set of commuting flows.

The Bordner-Corrigan-Sasaki embedding

We keep our representation space $V = \mathbb{C}^N$ of a sum of weight spaces \mathbb{C}_w , invariant under the Weyl group, and still have our embedding $\hat{\Xi}$, turning our H -bundles $E(h)$ into bundles $E_{GL(N)} = \hat{\Xi}(E(h)) = \oplus_i L_{w_i(h)}$. We now extend the embedding $\hat{\Xi}$ of the space of H bundles to $(T^*\mathcal{M}_G)_{\text{red}}$ in a different way, corresponding to the Lax pairs of Bordner-Corrigan-Sasaki [BCS2]. This involves the construction of a different $GL(N)$ Higgs field.

We first define some sections of $End(E_{GL(N)})$ associated to sections of the bundles $L_{\alpha(h)}$. We note that to each root α , we have an associated reflection $R_\alpha(v) = v - \langle \hat{\alpha}, v \rangle \alpha$ of the Lie algebra \mathfrak{h} , and in turn a permutation of the weight spaces \mathbb{C}_w , which can be represented by a matrix $(s_\alpha)_{w,w'} \in GL(V)$, where, as usual we index the entries of the matrix by the weights themselves. Note that the non-zero entries of s_α must have $w - w' = n\alpha$ for some integer n . For a section f of $L_{\alpha(h)}$, we define

$$\tilde{s}_\alpha(f)_{w,w'} = \sum_n (s_\alpha)_{w,w'} \delta_{w-w', n\alpha} n \cdot I_n^* f. \quad (6.26)$$

Represent the section ϕ , which is a section of the associated bundle $E((\oplus_\alpha \mathbb{C}_\alpha) \oplus \mathfrak{h})$ by $((\phi_\alpha), \phi_{\mathfrak{h}})$. We define the corresponding section $\phi_{GL(N)}$ is given by

$$\phi_{GL(N)} = ((\sum_\alpha \tilde{s}_\alpha(\phi_\alpha)) + \xi(\phi_{\mathfrak{h}})). \quad (6.27)$$

This section $\phi_{GL(N)}$ has poles not only at the origin, but also, for its $w, w + n\alpha$ components, at the n -th roots of unity in the curve Σ . The moduli space of $GL(N)$ Higgs pairs must be chosen accordingly.

The Hamiltonian is again a multiple of the Hamiltonian given by pairing $tr(\phi_{Gl(N)}^2)$ with our standard cocycle ω of (5.7); as above, the flow is given by (6.15)

$$\begin{aligned} M^{-1}\dot{M} &= (\omega\phi_{Gl(N)}^0)_d \\ (\dot{\phi}_{Gl(N)}^0) &= a + [(\omega\phi_{Gl(N)}^0)_{od}^0, (\phi_{Gl(N)}^0)] + [\omega\phi_{Gl(N)}^0{}_{od}{}^{\text{cst}}, (\phi_{Gl(N)}^0)]_d, \\ &= a + [(\omega\phi_{Gl(N)}^0)_{od}^0 + (\omega\phi_{Gl(N)}^0{}_{od}{}^{\text{cst}}, (\phi_{Gl(N)}^0)] - [(\omega\phi_{Gl(N)}^0{}_{od}{}^{\text{cst}}, (\phi_{Gl(N)}^0)]_{od}, \end{aligned} \quad (6.28)$$

with again the equation(6.16) for a .

Explicitly, one has

$$\begin{aligned} (\phi_{Gl(N)}^0)_{w,w'} &= \sum_{\alpha} \sum_n (s_{\alpha})_{w,w'} \delta_{w-w', n\alpha} n \rho^0(\alpha(x), nz), \\ ((\omega\phi_{Gl(N)}^0)_{od}^0 + (\omega\phi_{Gl(N)}^0{}_{od}{}^{\text{cst}})_{w,w'}) &= \sum_{\alpha} \sum_n (s_{\alpha})_{w,w'} \delta_{w-w', n\alpha} \frac{\partial \rho^0}{\partial x}(\alpha(x), nz). \end{aligned}$$

One again wants to use the action of the diagonal subgroup to write (6.28) as a Lax pair. We take $d'(x)$, to be diagonal, and get

$$\begin{aligned} M^{-1}\dot{M} &= (\omega\phi_{Gl(N)}^0)_d \\ (\dot{\phi}_{Gl(N)}^0) &= [(\omega\phi_{Gl(N)}^0)_{od}^0 + \omega\phi_{Gl(N)}^0{}_{od}{}^{\text{cst}} + d'(x), (\phi_{Gl(N)}^0)], \end{aligned} \quad (6.29)$$

In this case, the appropriate diagonal terms are given in [BCS2]:

$$d'(x)_{w,w} = \sum_{\alpha} (s_{\alpha})_{w,w} \frac{\partial \rho^0}{\partial x}(\alpha(x), 0). \quad (6.30)$$

In this case, there is no constraint on the representation; indeed, Bordner, Corrigan and Sasaki create a "universal Lax pair" within the algebra $\mathbb{C}(H) \otimes \mathbb{C}[W]$ created by tensoring the group algebra of the Weyl group with the function field of H ; the product must be suitably defined, but corresponds roughly to representing the Weyl group as acting by reflections on a sum of weight spaces, and the group H as acting by diagonal matrices. One then represents this algebra into a sum of weight spaces, and the Lax pair gets embedded into $Gl(N)$. This is what is given above. It would be interesting to do the geometry of bundles directly within this algebra.

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